III. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ABC be the given triangle, AB base, C the vertex. Let P be the given point without, $PA=b$, $\angle PAC=\theta$, $\angle BAC=A$, area of square$=m^2$, area of triangle$=\triangle$, $\triangle-m^2=n$. Let the line from P cutting off the area m^2 meet AC in D, AB in E; $AD=y$, $AE=x$.

Then $\frac{1}{2}xy\sin A=m^2$ ....(1).

$\frac{1}{2}by\sin\theta=\text{area}PAD$ ....(2).

$\frac{1}{2}bx\sin[\theta+A]=\text{area}PAE$ ....(3).

$\therefore \frac{1}{2}bx\sin[\theta+A]-\frac{1}{2}by\sin\theta=m^2$ ....(4).

The value of y from (1) in (4) gives

$$x^2-\frac{2m^2x}{b\sin[\theta+A]}=\frac{2m^2\sin\theta}{\sin A\sin[\theta+A]}.$$

$$\therefore x=\frac{m^2}{b\sin[\theta+A]}\pm\sqrt{\frac{2m^2\sin\theta}{\sin A\sin[\theta+A]}+\frac{m^4}{b^2\sin^2[\theta+A)}},$$

$$y=\frac{\sin[\theta+A]}{\sin\theta}\left[\pm\sqrt{\frac{2m^2\sin\theta}{\sin A\sin[\theta+A]}+\frac{m^4}{b^2\sin^2[\theta+A]}}-\frac{m^2}{b\sin[\theta+A]}\right].$$

If m^2 is to be the quadrilateral part of the triangle, substitute n for m^2 in the above values for x, y.

Two geometrical constructions which arrived too late for insertion will be printed next month.

219. On account of the pedagogical importance of this problem, the solutions are withheld until next month in order that they may all be printed in the same number.

CALCULUS.

Problem 173 was also solved by L. E. Newcomb, Los Gatos, California.

174. Proposed by B. F. FINKEL, A. M., 204 St. Marks Square, Philadelphia, Pa.

Integrate $\int_0^\infty \frac{\sinh px}{\sinh qx}\cos rx\,dx$, if $p^2<q^2$.

I. Solution by WILLIAM HOOVER, Ph. D., Professor of Mathematics, Ohio State University, Athens, O.

$$I=\int_0^\infty \frac{\sinh px}{\sinh qx}\cos rx\,dx=\int_0^\infty \frac{e^{px}-e^{-px}}{e^{qx}-e^{-qx}}\cos rx\,dx \quad\text{..........(1).}$$

In (1), put $qx=\pi z$, or $x=[\pi/q]z$; then (1) is

$$I=\frac{\pi}{q}\int_0^\infty \frac{e^{az}-e^{-az}}{e^{\pi z}-e^{-\pi z}}\cos mz\,dz \quad\text{..........(2),}$$

in which $a=p\pi/q$, $m=r\pi/q$, and the required integral

$$I = \frac{\pi}{q} \cdot \frac{\sin a}{e^m + 2\cos a + e^{-m}} \text{ .... } (3),$$

as given by Williamson's *Integral Calculus*, Appleton Edition of 1884, p. 143, Ex. 8, and by Carr's *Synopsis of Mathematics*, Edition of 1886, Equation 2594, p. 386. I am indebted to the latter work for the outline of the integral in equation (2). If we make, in $I' = \int_0^\infty \frac{e^{ax} - e^{-ax}}{e^{\pi x} - e^{-\pi x}} \sin mx\, dx$, the indicated division,

$$I' = \int_0^\infty [e^{ax} + e^{-ax}] \sin mx [e^{-\pi x} + e^{-3\pi x} + e^{-5\pi x} + \text{etc.}] dx,$$

in which, if we remove brackets, there occur terms of the type $\int_0^\infty e^{-a'x} \sin bx\, dx$, whose integral is, $\frac{b}{a'^2 + b^2}$, Williamson, Ex. 19, p. 117, and

$$I' = \sum \frac{m}{\{[2n-1]\pi - a'\}^2 + m^2} + \sum \frac{m}{\{[2n-1]\pi + a'\}^2 + m^2}.$$

We have, by Casey's *Treatise on Plane Trigonometry*, Ed. 1888, Ex. 33, p. 231,

$$\frac{\cos x + \cos a'}{1 + \cos a'} = \left[1 - \frac{x^2}{[\pi - a']^2}\right]\left[1 - \frac{x^2}{[\pi + a']^2}\right]\left[1 - \frac{x^2}{[3\pi - a']^2}\right]$$

$$\times \left[1 - \frac{x^2}{[3\pi + a']^2}\right]\left[1 - \frac{x^2}{]5\pi - a']^2}\right] \text{..............}$$

In this put $x = ix$, and then taking the derivative of the logarithm of both members, noting that $\cos ix = \frac{1}{2}[e^{-x} + e^x]$, we have

$$\frac{1}{2} \frac{e^x - e^{-x}}{e^x + 2\cos a' + e^{-x}} = \frac{x}{(\pi - a')^2 + x^2} + \frac{x}{(\pi - a')^2 + x^2}$$

$$+ \frac{x}{(3\pi - a')^2 + x^2} + \frac{x}{(3\pi + a')^3 + x^2} + \text{ etc.}$$

If in this we put $x = m$, and in the above summation for I' we make $n = 1, 2$, etc., we have, finally,

$$I' = \int_0^\infty \frac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}} \sin mx\, dx = \frac{1}{2} \frac{e^m - e^{-m}}{e^m + 2\cos a + e^{-m}}.$$

If in this we put $m = i\theta$, noting that $\sin(i\theta x) = \frac{1}{2}(e^{\theta x} - e^{-\theta x})$,

$$\int_0^\infty \frac{({}^{ax} + e^{-ax})(e^{\theta x} - e^{m\theta x})}{e^{\pi x} - e^{-\pi x}} dx = \frac{\sin\theta}{\cos a + \cos\theta}.$$

In this, change a into im, θ into a, and we have

$$\int_0^\infty \frac{e^{ax}-e^{-ax}}{e^{\pi x}-e^{-\pi x}}\cos mx\, dx=\frac{\sin a}{e^m+2\cos a+e^{-m}}.$$

Restoring the values of a and m, (3) becomes

$$I=\frac{\pi}{q}\frac{\sin p\pi/q}{e^{r\pi/q}+2\cos p\pi/q+e^{-r\pi/q}}.$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $x=\pi y/q$, $a=\pi p/q$, $b\sqrt{-1}=\pi r/q$.

$$A=\int_0^\infty \frac{\sinh px \cos rx}{\sinh qx}dx=\frac{\pi}{q}\int_0^\infty \frac{\sinh ay \cos b\sqrt{(-1)}y}{\sinh \pi y}$$

$$=\frac{\pi}{2q}\int_0^\infty \frac{(e^{ay}-e^{-ay})(e^{by}+e^{-by})}{e^{\pi y}-e^{-\pi y}}dy$$

$$=\frac{\pi}{2q}\int_0^\infty \left[\frac{e^{(a+b)y}-e^{-(a+b)y}}{e^{\pi y}-e^{-\pi y}}+\frac{e^{(a-b)y}-e^{-(a-b)y}}{e^{\pi y}-e^{-\pi y}}\right]dy$$

$$=\frac{\pi}{4q}\tan\tfrac{1}{2}(a+b)+\frac{\pi}{4q}\tan\tfrac{1}{2}(a-b).$$

(Williamson's *Integral Calculus*, p. 142).

$$\tan\tfrac{1}{2}(a+b)+\tan\tfrac{1}{2}(a-b)=\frac{2\sin a}{\cos a+\cos b}.$$

$$\therefore A=\frac{\pi}{2q}\cdot\frac{\sin a}{\cos a+\cos b}=\frac{\pi}{2q}\cdot\frac{\sin(\pi p/q)}{\cos(\pi p/q)+\cosh(\pi r/q)}.$$

175. Proposed by M. E. GRABER, A. B., Instructor in Mathematics and Physics in Heidelberg University, Tiffin, Ohio.

Find the volume of the cono-cuneus determined by $z^2+a^2y^2/x^2=c^2$, which is contained between the planes $x=0$ and $x=a$. Ans. $\frac{1}{2}\pi c^2 a$. [Todhunter's *Integral Calculus*, p. 189, No. 28].

I. Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

The section by a plane parallel to the yz plane is an ellipse whose semi-axes are c and cx/a, and whose area is therefore c^2x/a. Hence the required volume is $\frac{\pi c^2}{a}\int_0^a xdx$, or $\frac{1}{2}\pi ac^2$.